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A GENERALIZATION OF THE LERAY-SCHAUDER INDEX FORMULA.(U)
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A GENERALIZATION OF THE LERAY-SCHAUDER INDEX FORMULA

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ABSTRACT

This paper generalizes the Leray-Schauder index formula to the case where the inverse image of a point consists of a smooth manifold, assuming some nondegeneracy condition is satisfied on the manifold. The result states that the index is the Euler characteristic of a certain vector bundle over the manifold. Under slightly stronger nondegeneracy conditions, the index is in fact the Euler characteristic of the manifold.

The paper also includes a discussion of the Euler characteristic for vector bundles and a simple proof of the Gauss-Bonnet-Chern theorem.

AMS(MOS) Subject Classifications: 47G10, 47H15, 53A55.

Key Words: Leray-Schauder degree; Euler characteristic; Gauss-Bonnet-Chern Theorem.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The Leray-Schauder degree is one of the basic methods of nonlinear functional analysis. It is useful in proving existence theorems for many nonlinear differential and integral equations. The basic computational tool in the theory is the Leray-Schauder index formula, which allows one to compute the degree in a special case. This paper extends the computational formula to a more general setting.

The ideas used here are applied in the second part of the paper to prove in a very elementary way the Gauss-Bonnet-Chern theorem, a classical theorem in differential geometry.

A GENERALIZATION OF THE LERAY-SCHAUDER INDEX FORMULA

J. Sylvester

Introduction

Let F be a continuously differentiable mapping from an open subset Ω of a Banach space B into B. We assume that F has the form $I + K$ where I is the identity and K is a compact operator. In particular, this guarantees that for any $y \in B$, $F^{-1}(y)$ is a compact set. If $M \subset \Omega$ is any isolated component of $F^{-1}(y)$, we may define an integer, called the index, $i_F(M)$ by the formula:

$$i_F(M) = \deg(F, N_\varepsilon(M), y)$$

where $N_\varepsilon(M) = \{x \in \Omega \mid \text{dist}(x,M) < \varepsilon\}$ with ε chosen so small that $F^{-1}(y) \cap \overline{N_\varepsilon(M)} = M$, and $\deg(F, N_\varepsilon(M), y)$ is the Leray-Schauder degree. It is an immediate consequence of the definition and properties of the degree (see for example (1) or (5)) that $\deg(F, N_\varepsilon(M), y)$ is independent of ε under the above hypothesis.

The Leray-Schauder index formula computes the value of $i_F(M)$ in the special case that M is a point and $DF(M)$ is an isomorphism. In this case, according to the formula:

$$i_F(M) = \deg(DF(M), N_\varepsilon(M), 0) = (-1)^{\rho(DF(M))}$$

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where $\rho(L)$ = the algebraic number of eigenvalues of the linear operator L which are real and strictly negative. It follows from the assumption that $F = I + K$ that this number is finite for $L = DF(M)$.

In this paper, we generalize the index formula to the case where M is a connected smooth manifold, under the restriction that $\ker(DF(m)) = T_mM$ for all m in M. We show that

$$i_F(M) = \chi(\xi)$$

where $\begin{smallmatrix}\xi \\ \downarrow \\ M\end{smallmatrix}$ is the vector bundle with base space M and fibres ξ_m = B/Range $DF(m)$ and $\chi(\xi)$ is its Euler characteristic. We remark that we do not make any assumptions about the orientability of ξ. In general, ξ will not be oriented, but the total space $E(\xi)$ will always be oriented for bundles which arise from this construction.

We begin the paper by defining the Euler characteristic for vector bundles with oriented total space and make some remarks as to why this is the appropriate class of vector bundles for which the Euler characteristic (although not necessarily the Euler chomology class) is defined.

In the second section we state and prove the generalization of the index formula.

We conclude with a simple proof of the Gauss-Bonnet-Chern Theorem which makes use of an abstract version of the Gauss mapping and proceeds along lines similar to those followed in the proof of the index formula. The proof is analogous to that of Allendörfer (6) in the embedded case.

The author would like to express his gratitude to his thesis advisor, Professor Louis Nirenberg, both for suggesting this problem and for his constant interest and support throughout the period when this work was being done.

§1. The Euler Characteristic

Let $\begin{smallmatrix}\xi\\ \downarrow p\\ M\end{smallmatrix}$ be a smooth n-dimensional vector bundle with orientable total space E. We assume M is a compact smooth n-dimensional manifold without boundary. Suppose $\bar{s}$ is a section of ξ. If $\bar{s}(m) = (m, s(m))$, then for all $m \in Z = \{m \mid s(m) = 0\}$ $Ds(m): T_m M \to \xi_m$. By Sard's theorem, we may pick $\bar{s}$ such that Z is discrete and such that $Ds(m)$ is an isomorphism for all $m \in Z$. For each $z \in Z$, we pick a basis $\langle e_1 \ldots e_n \rangle$ for $T_z M$ and define the function

$$K(z) = \begin{cases} +1 & \text{if } \langle e_1,\ldots,e_n, Ds(z)e_1,\ldots,Ds(z)e_n\rangle \\ & \text{is a positively oriented for } T_z E \\ -1 & \text{if } \langle e_1,\ldots,e_n, Ds(z)e_1,\ldots,Ds(z)e_n\rangle \\ & \text{is a negatively oriented basis for } T_z E \end{cases}$$

then

$$\chi(\xi) = \sum_{z \in Z} K(z) \quad .$$

Note that $K(z)$ is independent of the choice of basis $\langle e_1,\ldots,e_n\rangle$; if

$$b_i = A_i^j e_j$$

is another basis for $T_z M$, we have

$$\langle b_1 \ldots b_n, Ds(z)b_1,\ldots Ds(z)b_n\rangle =$$

$$\begin{pmatrix} A & 0 \\ 0 & Ds(z)ADs(z)^{-1} \end{pmatrix} \langle e_1 \ldots e_n, Ds(z)e_1,\ldots Ds(z)e_n\rangle$$

and the matrix has positive determinant.

We sketch the proof that $\chi(\xi)$ is independent of the section $\bar{s}$. We begin with

<u>Lemma 1</u>. Let $\begin{smallmatrix}\xi\\ \downarrow\\ M\end{smallmatrix}$ be an n-dimensional vector bundle over a compact manifold M, then there exists an injective bundle mapping j

$$j : \begin{smallmatrix}\xi\\ \downarrow\\ M\end{smallmatrix} \to \begin{smallmatrix}\mathbf{R}^N \times M\\ \downarrow\\ M\end{smallmatrix}$$

for some N. Furthermore, if ξ has a Riemannian metric g we may choose j so that g is the pull back of the standard metric on $\mathbf{R}^N$ (this fact will be of use later).

<u>Proof</u>

Let (U_α, h_α) $\alpha = 1,\ldots,k$ be a trivializing cover for ξ, and let ϕ_α^2 be a subordinate partition of unity. Define

$$\mathbf{R}^N \times M = \left\{ \bigoplus_{\alpha=1}^{k} \mathbf{R}_\alpha^n \right\} \times M$$

and let $i_\alpha : M \times \mathbf{R}^n \to M \times \mathbf{R}^N$ be the obvious bundle embedding with range at each point $m \in M$ equal to $\mathbf{R}_\alpha^n$. Now define

$$j = \sum_{\alpha=1}^{k} \phi_\alpha i_\alpha \cdot h_\alpha^{-1} .$$

In order to obtain the metric g, we merely pick the h_α so that $g|_{p^{-1}(U_\alpha)}$ is the pull back of the standard metric on $U^\alpha \times \mathbf{R}^n$.

One easily verifies that j has the desired properties.

<u>The complementary bundle</u> η to ξ in $\mathbf{R}^N \times M$ is defined to be the quotient bundle $\mathbf{R}^N/\xi$. η is defined up to isomorphism by the property:

$$\xi \underset{M}{\oplus} \eta \cong R^N \underset{M}{\times} M \;.$$

The existence of the quotient bundle, as well as most of lemma 1, is standard and may be found in (3) or (2). We remark that if $E(\xi)$ is orientable, $E(\eta)$ is also.

Definition of the Gauss Mapping

Consider the following sequence of maps

$$\underset{M}{\eta} \xrightarrow{i} \xi \underset{M}{\oplus} \eta = R^N \times M \xrightarrow{\pi} R^N \tag{1}$$

where i is the obvious embedding and π projects onto the second factor. We define

$$G : E(\eta) \to R^N \text{ by } G = \pi \cdot i \;.$$

If we let $E_1(\eta) = \{(x,v) \in \eta \mid |\pi \cdot i(x,v)| < 1\}$, we have

Lemma 2. $\deg(G, E_1(\eta), 0) = \chi(\xi)$.

We do not include the proof, as it will appear (with a few cosmetic changes due to the Banach space context) in the proof of the index formula in the next section.

This finishes the proof that $\chi(\xi)$ is independent of $\bar{s}$ as $\deg(G, E_1(n), 0)$ is independent of any section.

The Euler characteristic and the Euler class

We remark that our assumption that $E(\xi)$ be oriented is different from that usually made in the literature. It is customary to assume that the vector bundle itself, not the total space, is oriented.

The two assumptions differ only in case the base manifold is non-orientable. For example, the tangent bundle of a non-orientable manifold is not orientable, although the Euler characteristic can be defined as the

alternating sum of the betti numbers or as the Lefschetz number of the identity map, both of which make sense without assuming orientability. It is immediate, however, that the total space of any tangent bundle is orientable.

It should be noted that for an oriented vector bundle we may define the Euler cohomology class, while in general no such integral class on the base manifold exists if we only assume that the total space is oriented. For our application in §2, it is the integral invariant which plays the central role and the existence of the cohomology class is not important.

Finally, we observe that one may construct a cohomology class, not on the base space, but on its two fold orientable covering space. The pull back bundle will always be orientable and the "Euler class" for the original bundle will be exactly half that of the pull back bundle.

§2. The Index Formula

Let B be Banach space, Ω an open subset of B and F a mapping satisfying

(i) $F : \Omega \subset B \to B$

(ii) $F = I + K$; K compact

(iii) $F \in C^1(\Omega)$

(iv) $M \subset F^{-1}(y) \subset \Omega$ is a connected smooth manifold and

nullity $(DF(m)) = \dim M \quad \forall\, m \in M$.

We remark that (ii) implies that M is both compact and finite dimensional. For F satisfying (i) - (iv) we have

$$i_F(M) = \deg(F, N_\varepsilon(M), y)$$

where $N_\varepsilon(M) = \{x \in B \mid \mathrm{dist}(x,M) < \varepsilon\}$ is a tubular neighborhood of M. We will show that the right hand side is defined and independent of ε for ε sufficiently small. We prove:

__Theorem__. Let F satisfy (i) - (iv), then $\exists\, \varepsilon_0 > 0$ such that $\forall\, \varepsilon < \varepsilon_0$

$$i_F(M) = \deg(F, N_\varepsilon(M), y) = \chi(\xi)$$

where ξ is the vector bundle with base M and fibre $\xi_x = B/\text{Range } DF(x)$ (the orientation of ξ will be described below) and χ is its Euler characteristic.

__Corollary__. If F satisfies (i) - (iv) and

(v) Range $DF(M) \cap \ker DF(M) = \{0\} \quad \forall\, m \in M$

then

$$i_F(M) = (-1)^{\rho(DF(m))}\chi(M).$$

__Proof of the Corollary__

We first note that (iv) and (v) fix the spectral multiplicity of zero for $DF(m)$ independent of $m \in M$; this in turn fixes $\rho(DF(m))$ modulo 2, so that the formula is independent of m. As a consequence of (v)

$$B = \text{Range } DF(m) \oplus \ker DF(m)$$

which implies $\xi = B/\text{Range } DF(m) \cong \ker DF(m) = T_m M$ where the explicit isomorphism is given by the projection operator $P(m)$ onto $\ker DF(m)$ along Range $DF(m)$. The necessary smoothness of $P(m)$ is easily verified from the integral formula

$$P = \int_\Gamma R_\gamma(DF(m))\,d\gamma$$

where Γ is a contour about zero and R_Γ denotes the resolvent ((v) guarantees that P is a spectral projection).

The factor of $(-1)^{\rho(DF(m))}$ provides for the appropriate orientation as will be described in the proof of the theorem.

<u>Proof of theorem</u>

I. $\deg(F, N_\varepsilon(M), y)$ is defined and independent of ε for all $0 < \varepsilon < \varepsilon_0$.

We observe that it follow from (iv) that for $x \in N_\varepsilon(M)$

$$\begin{aligned} F(x) &= F(m) + DF(m)(x-m) + o(|x-m|) \\ &= 0 \quad + DF(m)(x-m) + o(|x-m|) \end{aligned}$$

so that

$$|F(x)| > \tfrac{1}{2}\,|DF(m)(x-m)| \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad x \in N_\varepsilon(M)$$

and therefore $F(x)$ is nonzero for $x \in N_\varepsilon(M)\backslash M$. Hence $F|_{\partial N_\varepsilon(M)} \neq 0$ and $\deg(F, N_\varepsilon(M), y)$ is well defined; by the excision property of degree, it is independent of $\varepsilon > 0$.

II. Orientation of $E(\xi)$

We begin by introducing the complementary vector bundle[1] $\begin{array}{c}\eta\\ \downarrow\\ M\end{array}\pi_\eta$, where $\eta_m \subset B$ and $\eta_m \oplus \text{Range } DF(m) = B$ (η_m is the fibre over the point m). There exists a natural bundle isomorphism from η to ξ, namely the mapping which takes each vector to its equivalence class. Henceforth we shall deal with η and describe the orientation of $E(\eta)$ as follows:

(1) As $T_{(m,v)}E(n)$ is naturally isomorphic to $T_mM \oplus \eta_m$ a frame in $T_mM \oplus \eta_m$ of the form $\langle w_1 \ldots w_m, v_1 \ldots v_m\rangle$, where $\langle w_1,\ldots,w_n\rangle$ spans T_mM and $\langle u_1,\ldots,u_n\rangle$ spans η_m, defines an isomorphism

$$O_m(v,w) : T_mM \to \eta_m$$

by the formula

$$O_m(v,w)w_i = v_i \quad .$$

(2) Let $\begin{array}{c}\nu\\ \downarrow\\ m\end{array}$ be a complementary bundle[1] to TM and let P_m be the projection onto T_mM along ν_m, then the linear isomorphism

$$DF(m) + O_m(v,w)P_m : B \to B$$

has the form $I + K$, K compact, and hence has degree plus or minus one. We say that $\langle w_1,\ldots,w_n, v_1,\ldots,v_n\rangle$ is positively oriented if the degree of the map is plus one.

To check that this defines a global orientation we merely note that $\langle v_1,\ldots,v_m, w_1,\ldots,w_m\rangle$ may be extended to local sections of $TE(\eta)$ and as $DF(m) + O_m(v,w)$ remains (locally) an isomorphism, its degree remains $+1$.

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(1) For any subbundle of B with finite dimension or codimension, the existence of a complementary bundle follows from the Hahn-Banach theorem. This bundle is only unique up to isomorphism.

III. Computation of $i_F(M)$

Let s be a mapping from M into B such that $s(x) \in \eta_x$ for all $x \in M$ (i.e. s is a smooth section of η). By Sard's theorem, we may choose s to have isolated and nondegenerate zeroes (i.e. $D_s(x)$ should have full rank for all x such that $s(x) = 0$). As above $\nu_m \xrightarrow{} p$ is the complementary bundle to TM and p projects onto the base point. It is the content of the tubular neighborhood theorem that the mapping $(m,v) \xrightarrow{i} m + v$ is a diffeomorphism from ν_ε to $N_\varepsilon(m)$. We denote by π the mapping $p \cdot i^{-1}$ (2).

Finally, define $\tilde{F}(x) = DF(\pi(x))\,(x - \pi(x))$. We now prove:

<u>Lemma</u> $\deg(F, N_\varepsilon(M), 0) = \deg(\tilde{F} + s \cdot \pi, N_\varepsilon(M), 0)$.

<u>Proof</u>. We expand F as a Taylor polynomial about points in M:

$$\begin{aligned} F(x) &= F(\pi(x)) + DF(\pi(x))\,(x - \pi(x)) + o(|x - \pi(x)|) \\ &= \quad 0 \quad + DF(\pi(x))\,(x - \pi(x)) + o(|x - \pi(x)|) \end{aligned}$$

which implies that for ε small the homotopy

$$G(t,x) = tF(x) + (1-t)DF(\pi(x))\,(x - \pi(x))$$

satisfies $|G(t,x)| \geq |DF(\pi(x))\,(x - \pi(x))| - o(|x - \pi(x)|)$ and by (iv), $|DF(\pi(x))\,(x - \pi(x))| \geq C|x - \pi(x)|$ where C is independent of x as M is compact. This shows that $G|_{\partial N_\varepsilon(M)} \neq 0$ and hence that

$$\deg(F, N_\varepsilon(M), 0) = \deg(\tilde{F}, N_\varepsilon(M), 0) \quad .$$

Similarly, we define the homotopy

$$H(t,x) = \tilde{F}(x) + ts(\pi(x)) \quad .$$

The observation that $s(x) \in \eta_x$ where Range $\tilde{F} \oplus \eta = B$ implies that

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(2) In the case B is a Hilbert space, we may take π to be simply the map which associates to each point in $N_\varepsilon(M)$ the closest point in M. This map is of course smooth if ε is sufficiently small.

$$H(t,x) = 0 \text{ if and only if } \begin{cases} \tilde{F}(x) = 0 \\ \text{and } s(\pi(x)) = 0 \end{cases} .$$

In particular, $\tilde{F}(x) = 0$ only when $x \in M$ and therefore not on $\partial N_\varepsilon(M)$, so that $H|_{\partial N_\varepsilon(M)} \neq 0$ and the lemma is established.

We now compute $\deg(\tilde{F} + s \circ \pi, N_\varepsilon(M), 0)$ using the Leray-Schauder formula. By the last remark in the proof of the lemma, we see that $\tilde{F} + s \circ \pi$ vanishes only on M and further, that it vanishes exactly at the zeroes of s. At these zeroes $x = \pi(x)$, so that

$$\begin{aligned} D_w(\tilde{F} + s \circ \pi) &= D_w\left[DF(\pi(x))\,(x - \pi(x)) + s(\pi(x))\right] \\ &= D^2F(\pi(x))\,(D\pi(x)w,\ x - \pi(x)) + \\ &\quad + DF(\pi(x))\,(I - D\pi(x))w + Ds(\pi(x))D\pi(x)w \\ &= DF(x)\,(I - P_x))w + Ds(x)P_x w \end{aligned}$$

where $P_x = D\pi(x)$ is the projection onto T_xM along ν_x. Finally, as $DF(x)P_x = 0$, we have

$$D_w(\tilde{F} + s \circ \pi) = (DF(x) + Ds(x)P_x)w .$$

By Leray-Schauder,

$$\begin{aligned} \deg(\tilde{F} + s \circ \pi, N_\varepsilon(M), 0) &= \sum_{x \in v^{-1}(0)} \deg(DF(x) + Ds(x)P_x) \\ &= \sum_{x \in v^{-1}(0)} i_v(x) \equiv \chi(\eta) . \end{aligned}$$

The last step being justified as the $\deg(DF(x) + Ds(x)P_x)$ is $+1$ exactly when $\langle v_1 \ldots v_n, Dsv_1 \ldots Dsv_n \rangle$ is positively oriented and -1 when this frame is negatively oriented.

§3. A Proof of the Gauss-Bonnet-Chern Theorem

Theorem. (Gauss-Bonnet-Chern) Let ξ be a 2m-dimensional Riemanian vector bundle with metric g, compatible connection ∇, and associated curvature matrix (in some orthonormal frame) Ω. The n-form $L(\xi) = L(\nabla,\xi)$ defined by

$$L(\xi) = Pf(\Omega) = \frac{1}{2m!}\sum_I \varepsilon^I \Omega_{i_1 i_2}\ \Omega_{i_3 i_4} \cdots \Omega_{i_{2m-1} i_{2m}}$$

has the property that

$$\int_M L(\xi) = \frac{\pi^m m! 2^{2m}}{(2m)!}\chi(\xi) \quad .$$

(ε^I is the sign of the permutation $[1,2,\ldots,2m] \to [i_1,i_2,\ldots,i_{2m}]$.

Proof

We begin by assuming that M is orientable, if not $L(\xi)$ must be zero as a nonorientable manifold cannot support a nonzero n-dimensional integral cohomology class. The orientation of M, along with that of the bundle ξ, gives an orientation on $E(\xi)$ and we may compute $\chi(\xi)$ From Lemma 2 of §1.

Specifically, if we let $\theta^1,\ldots\theta^N$ be coordinates on R^N, we have

$$\chi(\xi) = \deg(G, E_1(\eta), 0) = \frac{\int_{E_1(\eta)} G^*(d\theta^1 \wedge \cdots \wedge d\theta^N)}{\int_{B^N} d\theta^1 \wedge \cdots \wedge d\theta^N} \qquad (2)$$

where B^N is the ball of radius one in R^N (see (1)). We define a 2m-form $X(\xi)$ on M by

$$X(\xi) = \frac{\int_{\eta_x} G^*(d\theta^1 \wedge \cdots \wedge d\theta^N)}{\int_B d\theta^1 \wedge \cdots \wedge d\theta^N}$$

where $\int_{\eta_x}$ means integration over the positively oriented fibre. It is immediate from (2) that

$$\int_M X(\xi) = \chi(\xi) \quad .$$

We shall prove the theorem by explicitly calculating $X(\xi)$ and showing that

$$X(\xi) = L \quad (\nabla, \xi) \tag{3}$$

where ∇ is the connection on ξ obtained by pulling back the flat connection on $M \times \mathbb{R}^N$. ∇ is obviously compatible with the pull back metric, which we can arrange to be any metric we wish by Lemma 1 of §1. To establish the general theorem, we then quote the following simple lemma.

<u>Lemma 3</u> $\int_M L(\nabla, \xi)$ is independent of ∇, provided ∇ is compatible with g.

<u>Proof</u>

Given ∇_1 and ∇_2, one constructs the family of connections $\nabla_t = t\nabla_1 + (1-t)\nabla_2$, which are compatible with g, and the n-forms $L(\nabla_t, \xi)$, which provide a homotopy from $L(\nabla_1, \xi)$ to $L(\nabla_2, \xi)$. (See (3) or (4) for more details.)

We proceed to calculate $X(\xi)$; we shall need the following formula, the proof of which we omit.

<u>Lemma 4</u>. Let $\ell_1 \ldots \ell_q$ be integers, $\sum_{p=1}^{q} \ell_p = n$, where n is an even integer, then

$$\int_{|z|<1} z_1^{\ell_1} \cdots z_q^{\ell_q} dz_1 \ldots dz_q = \begin{cases} 0 \quad \text{if any } \ell_p \text{ odd} \\ \dfrac{(n/2)!}{n!} \dfrac{\ell_1!, \ldots, \ell_q!}{(\frac{\ell_1}{2})! \ldots (\frac{\ell_q}{2})!} \pi^{\frac{q-1}{2}} \Gamma(\frac{n+1}{2}) \\ \text{if all } \ell_p \text{ even} \end{cases} .$$

Let $e_1 \ldots e_N$ be an orthonormal basis for $\mathbb{R}^N$

$\theta^1 \ldots \theta^N$ the dual basis

Let $b_1(x) \ldots b_n(x)$ be a local orthonormal basis for ξ

$\beta^1 \ldots \beta^n$ the dual basis

Let $b_{n+1}(x) \ldots b_N(x)$ be a local orthonormal basis for η

$\beta^{n+1}(x) \ldots \beta^N(x)$ the dual basis

Finally, let $b_\alpha(x) = a_\alpha^\delta(x) e_\delta \qquad \alpha = 1, \ldots, N$. Note that $a_\alpha^\delta(x)$ is unitary, as both bases are orthogonal. We have

$$d\theta^1 \wedge \ldots \wedge d\theta^N = \bigwedge_{\alpha=1}^{N} d(a_\delta^\alpha \beta^\delta)$$

$$= \bigwedge_{\alpha=1}^{N} a_\alpha^\gamma d(a_\delta^\gamma \beta^\delta) \quad \text{because} \quad \det(\alpha_\alpha^\gamma(x)) = 1$$

$$= \bigwedge_{\alpha=1}^{N} (a_\alpha^\gamma a_\delta^\gamma d\beta^\delta + a_\alpha^\gamma da_\delta^\gamma \beta^\delta)$$

$$= \bigwedge_{\alpha=1}^{N} (d\beta^\alpha + (a_\alpha^\delta da_\delta^\gamma)\beta^\delta)$$

$$= \bigwedge_{\alpha=1}^{N} (d\beta^\alpha + \omega_{\alpha\delta}\beta^\delta)$$

where $\omega_{\alpha\delta} = a_\alpha^\gamma da_\delta^\gamma$ = the connection matrix for the flat connection on $\mathbb{R}^N \times M$ in the basis b_α.

$$G^*(d\theta^1 \wedge \dots \wedge d\theta^N) = \bigwedge_{i=1}^{n} (\omega_{ip}\beta^p) \bigwedge_{q=n+1}^{N} (d\beta^q + \omega_{qp}\beta^p)$$

where the sums on the index p range from $n+1$ to N. (We have used the fact that $G^*(\beta^j) = 0$ for $j = 1,\dots,n$.) We integrate

$$\int_{\eta_x} G^*(d\theta^1 \wedge \dots \wedge d\theta^N) = \int_{\eta_x} \bigwedge_{i=1}^{n} (\omega_{ip}\beta^p) \bigwedge_{q=n+1}^{N} d\beta^q$$

and expand the product on the right to obtain

$$\int_{\eta_x} G^*(d\theta^1 \wedge \dots \wedge d\theta^N) = \sum_{i \in Z_q^n} \omega_{1i_1} \wedge \dots \wedge \omega_{ni_n} \int \beta^{i_1} \dots \beta^{i_n} d\beta^{n+1} \wedge \dots \wedge d\beta^N$$

where Z_q^n is the set of all n-tuples $i = (i_1,\dots,i_n)$ with $1 \leq i_j \leq q$ for $j = 1,\dots,n$. For $1 \leq p \leq q$ we define $\ell_p = \ell_p(i) = \#\{k \mid i_k = p\}$ and let Z_{qo}^n be the set of $i \in Z_q^n$ such that ℓ_p is even for all p. By lemma 4,

$$\int_{\eta_x} G^*(d\theta^1 \wedge \dots \wedge d\theta^N)$$

$$= \frac{\pi^{\frac{q-1}{2}} \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+q+2}{2})} \sum_{i \in Z_{qo}^n} \omega_{1i_1} \wedge \dots \wedge \omega_{ni_n} \frac{(\frac{n}{2})!}{n!} \frac{\ell_1! \dots \ell_q!}{(\frac{\ell_1}{2})! \dots (\frac{\ell_q}{2})!} .$$

Dividing both sides by $\int_B d\theta^1 \wedge \dots \wedge d\theta^N = \dfrac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N+2}{2})}$ we obtain

$$X(\xi) = \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \sum_{i \in Z_{qo}^n} \omega_{1i_1} \wedge \cdots \wedge \omega_{ni_n} \frac{(\frac{n}{2})!}{n!} \frac{\ell_1! \cdots \ell_q!}{(\frac{\ell_1}{2})! \cdots (\frac{\ell_q}{2})!}$$

$$= \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \cdot \frac{1}{n!} \sum_{\pi \in S^n} \sum_{i \in Z_{qo}^n} \omega_{1i_{\pi(1)}} \wedge \cdots \wedge \omega_{ni_{\pi(n)}} \frac{(\frac{n}{2})!}{n!} \frac{\ell_1! \cdots \ell_q!}{(\frac{\ell_1}{2})! \cdots (\frac{\ell_q}{2})!}$$

where the previous step is justified by the observation that, for any $\pi \in S^n$, the number $\frac{\ell_1! \cdots \ell_q!}{(\frac{\ell_1}{2})! \cdots (\frac{\ell_q}{2})!}$ is the same for i and $i \cdot \pi$ (S^n is the permutation group).

$$X(\xi) = \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \frac{1}{n!} \sum_{i \in Z_{qo}^n} \frac{(\frac{n}{2})!}{n!} \frac{\ell_1! \cdots \ell_q!}{(\frac{\ell_1}{2})! \cdots (\frac{\ell_q}{2})!} \cdot$$

$$\cdot \left(\sum_{\pi \in S^n} \omega_{1i_{\pi(1)}} \wedge \cdots \wedge \omega_{ni_{\pi(n)}} \right) .$$

By a simple counting argument

$$X(\xi) = \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \frac{1}{n!} \sum_{i \in Z_q^{\frac{n}{2}}} \left(\sum_{\pi \in S^n} \omega_{1j_{\pi(1)}} \wedge \cdots \wedge \omega_{nj_{\pi(n)}} \right)$$

where $j_{2k} = j_{2k-1} = i_k$. Interchanging sums and reordering the permutations, we have

$$X(\xi) = \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \frac{1}{n!} \sum_{\pi \varepsilon S^n} \varepsilon^\pi \sum_{i \varepsilon Z^{\frac{n}{2}}} \omega_{\pi(1)j_1} \wedge \cdots \wedge \omega_{\pi(n)j_n}$$

where ε^π is the sign of the permutation

$$X(\xi) = \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \frac{1}{n!} \sum_{\pi \varepsilon S^n} \varepsilon^\pi (\omega_{\pi(1)p} \omega_{\pi(2)p}) \wedge \cdots \wedge (\omega_{\pi(n-1)p} \omega_{\pi(n)p})$$

where we again have an implied sum on $p = n+1, \ldots, N$. On recalling that $\Omega_{ij} = \omega_{ip} \omega_{jp}$

$$X(\xi) = \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} \frac{1}{n!} \sum_{\pi \varepsilon S^n} \varepsilon^\pi \Omega_{\pi(1)\pi(2)} \wedge \cdots \wedge \Omega_{\pi(n-1)\pi(n)}$$

$$= \frac{n!}{\pi^{\frac{n}{2}} 2^n (\frac{n}{2})!} L(\xi) \quad .$$

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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This paper generalizes the Leray-Schauder index formula to the case where the inverse image of a point consists of a smooth manifold, assuming some nondegeneracy condition is satisfied on the manifold. The result states that the index is the Euler characterisitc of a certain vector bundle over the manifold. Under slightly stronger nondegeneracy conditions, the index is in fact the Euler characteristic of the manifold.

The paper also includes a discussion of the Euler characteristic for vector bundles and a simple proof of the Gauss-Bonnet-Chern theorem.

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